

# Statistical Mechanical Theory of a Closed Oscillating Universe

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The date of receipt and acceptance will be inserted by the editor

**Abstract** Based on Newton's laws reformulated in the Hamiltonian dynamics combined with statistical mechanics, we formulate a statistical mechanical theory supporting the hypothesis of a closed oscillating universe. We find that the behaviour of the universe as a whole can be represented by a free entropic oscillator whose lifespan is nonhomogeneous, thus implying that time is shorter or longer according to the state of the universe itself given through its entropy. We conclude that time reduces to the entropy production of the universe and that a nonzero entropy production means that local fluctuations could exist giving rise to the appearance of masses and to the curvature of the space.

**Key words** nonequilibrium statistical mechanics-irreversibility- cosmological theories

## 1 Introduction

Cosmology, as an important discipline of scientific knowledge, tries to address fundamental questions beyond the realm of sciences and influencing our philosophical and even religious comprehension of the world. Perhaps the major question posed by cosmology is the origin of our universe. Nowadays, the most important theory providing an answer to this question is the Big-Bang theory [1]. Aside from others details, we are mainly interested here in one of the implications of the Big-Bang theory, i. e., "that the universe will one day end", [1]. According to a current speculation, in a closed universe, after the expansion eventually stops, a contraction will follow leading to an implosion into a singularity a process known as the Big Crunch. This oscillating behavior of the universe has important implications over

the concept of time, first because it is not clear that the initial and final states of the universe can be correctly conceptualized through the notion of time, understood as a fundamental quantity, and secondly because this behavior suggests time is created and destroyed at the beginning and the end of the universe.

In contrast with these implications, we know that in all fundamental physical theories time plays a central role. Mechanics, quantum mechanics and relativistic theories require time as a quantity through which the evolution of the system is conceptualized. However, thermodynamics does not require the concept of time in order to establish the "direction" in which spontaneous natural processes occur, that is, towards the state of thermodynamic equilibrium. These considerations lead to the following question: Is the time a fundamental quantity to describe the evolution of a system during a spontaneous natural process?

Beside the Big Bang theory other alternatives with respect to the origin and nature of the universe have been posed: is it open or closed...? it is cyclic?, or is there an arrow of time related with entropy and irreversibility?. Therefore these questions are still a recurrent topic of debate [2]-[10].

Here, we present a cosmological statistical mechanics theory of a closed universe not in equilibrium, oscillating between two thermodynamic equilibrium states which runs parallel to other cosmological models proposing an oscillating or cyclic universe [3], [4], [10]. Our statistical mechanical theory might complement these models. We assume that physical interactions give rise to time and space, that is, we consider that if physical interactions did not exist, time and space would not exist either. Our theory provides us with an expression of the entropy production of the universe based on first principles from which one infers that time is inhomogeneous and reduces to the production of entropy. Moreover, our theory implies that there is no global time arrow although it explains the existence of irreversible local phenomena [11] as the corresponding ones in a nonequilibrium fluid. A more general statistical theory can be elaborated by incorporating quantum and relativistic effects through a description in terms of the set of reduced density operators instead of the distribution vector in addition to the generalized von Neumann equation and the generalized von Neumann entropy. Density operators will account for quantum and also relativistic effects since the metrics of the space-time will also be contained in this description [12]. However, since the stochastic dynamic is determined by the eigenvalues of the Hamiltonian, this more mathematically elaborate theory will essentially lead to the same conclusions.

To mathematically formalize the previous statements, we will consider an isolated system composed by  $N$  particles or bodies (the closed universe) whose large-scale structure is governed by long-range interactions falling into the Hamiltonian of the universe itself.

In classical mechanics the state of an  $N$ -body system at any time is given by a set of  $3N$  generalized coordinates  $q_1, \dots, q_{3N}$  and  $3N$  conjugated generalized momenta  $p_1, \dots, p_{3N}$ . The value of these  $6N$  variables defines

a point in the  $6N$ -dimensional phase space  $\mathbf{\Gamma}$  corresponding to the system. This representative point of the system moves in the phase space along a trajectory determined by Hamilton's equations

$$\dot{q}_l = \frac{\partial H}{\partial p_l} , \dot{p}_l = -\frac{\partial H}{\partial q_l} , \quad (1)$$

where  $H$  is the Hamiltonian of the universe. Therefore, Hamilton's equations describe a flow in  $\mathbf{\Gamma}$  space whose density flow  $F(\mathbf{\Gamma}, t)$  varies according to the Liouville equation

$$i \frac{\partial}{\partial t} F = L F \quad (2)$$

with

$$L = i \{H, ..\} = i \sum_l \left( \frac{\partial H}{\partial q_l} \frac{\partial}{\partial p_l} - \frac{\partial H}{\partial p_l} \frac{\partial}{\partial q_l} \right) \quad (3)$$

being the Liouville operator. The description in terms of the Liouville equation is completely equivalent to that in terms of Hamilton's equations [13]. If the system occupies a volume  $V$  in the configurational space the density will vanish out of this volume, a fact which can be used to show that the Liouville operator is Hermitian [14].

Given an initial density  $F(\mathbf{\Gamma}, 0)$ , the formal solution of the Liouville equation (2) can be written

$$F(\mathbf{\Gamma}, t) = e^{-itL} F(\mathbf{\Gamma}, 0). \quad (4)$$

Since  $L$  is Hermitian, all its eigenvalues are real[15], which according to Eq. (4) implies that  $F(\mathbf{\Gamma}, t)$  will have an oscillatory behavior. This is an important general result based on the fully microscopic Hamilton dynamics of the system which allows a rigorous formulation a closed universe in terms of the Liouville equation (2). A consequence of the Hermitian character of the Liouville operator is that one may formulate a statistical mechanical theory of a closed oscillating universe in which the evolution is parametrized by means of the entropy of the universe itself, and in turn time is parametrized through the rate of variation of the entropy, i. e., the entropy production.

The paper is organized as follows. In section 2, we formulate the Generalized Liouville equation and give its formal solution. In section 3, we postulate the nonequilibrium entropy and obtain the entropy production of the universe itself. Section 4 is devoted to the derivation of the equations of the entropic oscillator corresponding to the universe and finally in section 5 we discuss our main conclusions.

## 2 Generalized Liouville equation

Let us consider that the Hamiltonian of the system is given by

$$H = \sum_{j=1}^N \frac{\mathbf{p}_j^2}{2m_j} + \frac{1}{2} \sum_{j \neq k=1}^N \phi(|\mathbf{q}_j - \mathbf{q}_k|) , \quad (5)$$

with  $m_j$  being the mass of a particle,  $\mathbf{q}_j$  the position vector of the  $j$ -th particle and  $\mathbf{p}_j$  its conjugated momentum. Moreover  $\phi(|\mathbf{q}_j - \mathbf{q}_k|) \equiv \phi_{j,k}$  is the interaction potential. The complete statistical description of the system can be given in terms of the distribution vector  $\mathbf{f}$  [13], [11]

$$\mathbf{f}(t) \equiv \{f_o, f_1(x_1, t), f_2(x^2, t), \dots, f_N(x^N, t)\} \quad (6)$$

which is the set of all the  $n$ -particle reduced distribution functions,

$$f_n = \int F(x^N, t) dx_{n+1} \dots dx_N, \quad (7)$$

where  $x_j = (\mathbf{q}_j, \mathbf{p}_j)$  and  $x^n = \{x_1, \dots, x_n\}$ ,  $n = 0, \dots, N$ . Here,  $f_o = 1$  and  $f_N(x^N) = F(x^N, t)$ . The dynamics of the distribution vector  $\mathbf{f}$  is obtained by integrating the Liouville equation (2) according to the definition of the  $n$ -particle reduced distribution functions (7), which gives

$$\frac{\partial}{\partial t} f_n = \{H_n, f_n\} + (N - n) \sum_{j=1}^n \int \mathbf{F}_{j,n+1} \frac{\partial}{\partial \mathbf{p}_j} f_{n+1} dx_{n+1}. \quad (8)$$

Here,  $\mathbf{F}_{j,n+1} = -\nabla_j \phi_{j,n+1}$  and

$$H_n = \sum_{i=1}^n \frac{\mathbf{p}_i^2}{2m_i} + \frac{1}{2} \sum_{i \neq k=1}^n \phi_{i,k} \quad (9)$$

is the  $n$ -particle Hamiltonian. The first term on the right hand side of Eq. (8) represents a Hamiltonian flow, while the second, a non-Hamiltonian contribution due to the coarse graining of the description. The set of equations represented by Eq. (8) constitutes a hierarchy of coupled equations, the Bogoliubov, Born, Green, Kirkwood, Yvon (BBGKY) hierarchy of equations [13], [11] representing the full microscopic description of the system. The BBGKY hierarchy, which can be represented in a compact way

$$i \frac{\partial}{\partial t} \mathbf{f}(t) = \mathcal{L} \mathbf{f}(t), \quad (10)$$

constituting the Generalized Liouville equation. By comparison of Eqs. (8) and (10) it can be seen that the Generalized Liouville operator  $\mathcal{L}$  splits into a diagonal Hermitian part  $\mathcal{P}\mathcal{L}$  defined through [13], [11]

$$\langle n | \mathcal{P}\mathcal{L} | n' \rangle = i [H_n, f_n]_P \delta_{n',n}, \quad n > 0, \quad (11)$$

where  $|n\rangle$  represents the  $n$ -particle state, and a nondiagonal non-Hermitian part  $\mathcal{Q}\mathcal{L}$

$$\langle n | \mathcal{Q}\mathcal{L} | n' \rangle = i \left\{ (N - n) \sum_{j=1}^n \int \mathbf{F}_{j,n+1} \frac{\partial}{\partial \mathbf{p}_j} f_{n+1} dx_{n+1} \right\} \delta_{n',n+1}, \quad n > 1. \quad (12)$$

Therefore, Eq. (10) can be rewritten

$$i \frac{\partial}{\partial t} \mathbf{f}(t) - \mathcal{P} \mathcal{L} \mathbf{f}(t) = \mathcal{Q} \mathcal{L} \mathbf{f}(t) \quad (13)$$

which explicitly manifests the Hermitian and non-Hermitian contributions to the dynamic of  $\mathbf{f}(t)$ . In Eq. (13), the term  $\mathcal{P} \mathcal{L} \mathbf{f}(t)$  plays the same role as the Liouville operator in Eq. (2) and since they are both Hermitian, they induce comparably incompressible flows. On the other hand, the term  $\mathcal{Q} \mathcal{L} \mathbf{f}(t)$  introduces long range correlations in the dynamics, which are related to the effects of coarse-grained interactions, and thus to dissipative effects in a reduced description. This implies that Eq. (13) provides us with more information than Eq. (2). In fact, as we will see, the non-Hermitian term is responsible for the approach to equilibrium, constituting a proof of the adequate description in terms of  $\mathbf{f}(t)$  instead of the full phase-space distribution function  $F$ .

The formal solution of Eq. (13) is

$$\mathbf{f}(t) = \exp(-i \mathcal{P} \mathcal{L} t) \mathbf{f}(0) + \exp(i \mathcal{P} \mathcal{L} t) \int_0^t d\tau \exp(-i \mathcal{P} \mathcal{L} \tau) (-i \mathcal{Q} \mathcal{L}) \mathbf{f}(\tau) , \quad (14)$$

which, proceeding iteratively, reduces to

$$\mathbf{f}(t) = \mathcal{U}(t, 0) \mathbf{f}(0) , \quad (15)$$

where the evolution operator  $\mathcal{U}(t, 0)$  is given by a perturbative development as

$$\begin{aligned} \mathcal{U}(t, 0) = & \sum_{j=0}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \dots \int_0^{t_{j-1}} dt_j \times \\ & \mathbf{V}(t, t_1) \dots \mathbf{V}(t_{j-1}, t_j) \exp(-i \mathcal{P} \mathcal{L} t_j) . \end{aligned} \quad (16)$$

Here,  $\mathbf{V}(t_{j-1}, t_j) = \exp[i \mathcal{P} \mathcal{L}(t_{j-1} - t_j)] (-i \mathcal{Q} \mathcal{L})$  are nonHermitian propagators,  $t_j < t_{j-1} < \dots < t_1 < t_0 = t$ , and the integration proceeds from right to left. Since  $\mathcal{P} \mathcal{L}$  is Hermitian, all its eigenvalues are real [15], which means that  $\mathbf{f}(t)$  as given through Eqs. (15) and (16) will have an oscillatory behavior. This oscillatory behavior has strong consequences in the evolution of the closed system, as we will show in the next sections.

### 3 Nonequilibrium entropy

In order to establish a connection between the N-particle microscopic description given by the generalized Liouville equation (13) and a macroscopic coarse-grained description, it is necessary to consider the statistical definition of the entropy. To achieve this objective, we start by noticing that the Gibbs entropy

$$S_N = -k_B \text{Tr}(F \ln F) = -k_B \int F \ln F dx^N , \quad (17)$$

where  $k_B$  is the Boltzmann constant, is a constant of motion under the Liouville dynamics given through Eq. (2), hence alternatively, as the nonequilibrium entropy for the N-body system, we propose [11]

$$\begin{aligned} S &= -k_B \text{Tr} \{ \mathbf{f} \ln \mathbf{f}_{eq}^{-1} \mathbf{f} \} + S_{eq} \\ &= -k_B \sum_{n=1}^N \int f_n \ln \frac{f_n}{f_{eq,n}} dx_1 \dots dx_n + S_{eq} , \end{aligned} \quad (18)$$

a convex functional of the distribution vector which generalizes the Gibbs formula. In Eq. (18),  $S_{eq}$  is the thermodynamic entropy (*i.e.* the equilibrium entropy) and  $\mathbf{f}_{eq}$  is the equilibrium distribution vector satisfying  $\mathcal{L}\mathbf{f}_{eq} = 0$ , the Yvon-Born-Green (YBG) equilibrium hierarchy[13]. Therefore,  $\mathbf{f}_{eq}$  is an eigenfunction of  $\mathcal{L}$  with eigenvalue 0.

From the convexity of the logarithmic function, it has been proven [16], [11] that the entropy  $S$  varies between the bounds determined through

$$0 \geq S_{eq} \geq S \geq S_N + S_{eq} . \quad (19)$$

Making use of Eq. (10) we can obtain the rate of change of  $S$ , which is the entropy production

$$\begin{aligned} \frac{\partial S}{\partial t} &= ik_B \text{Tr} \{ \mathcal{L} \mathbf{f} \ln (\mathbf{f}_{eq}^{-1} \mathbf{f}) \} = \\ &= -\frac{1}{T} \sum_{n=1}^N \sum_{j=1}^n \int f_n \mathbf{p}_j \left( -k_B T \frac{\partial}{\partial \mathbf{q}_j} \ln f_{eq,n} + \sum_{j \neq i=1}^n \mathbf{F}_{j,i} + (N-n) \mathcal{F}_j \right) dx^n . \end{aligned} \quad (20)$$

Here,  $\mathcal{F}_j(x^n)$  is the force on the  $j$ -th particle from the  $N-n$  particles not contained in the cluster of size  $n$ , and is defined through the relation:  $f_n(x^n) \mathcal{F}_j(x^n) = \int \mathbf{F}_{j,n+1} f_{n+1} dx_{n+1}$ , and  $T$  is the kinetic temperature obtained by taking into account that the dependence of  $f_{eq,n}$  on the velocities is given through a local Maxwellian. The entropy production given in Eq. (20) vanishes at equilibrium and in any other case it should not be necessarily zero. In addition, because  $\mathbf{p}_j$  is arbitrary,

$$\sum_{j \neq i=1}^n \mathbf{F}_{j,i} + (N-n) \mathcal{F}_j^{eq} = k_B T \frac{\partial}{\partial \mathbf{q}_j} \ln f_{eq,n} \quad (21)$$

is sufficient to satisfy the extremum condition  $\delta \dot{S} / \delta f_n|_{eq} = 0$ , with  $\dot{S} = \partial S / \partial t$ . Precisely, Eq. (21) gives rise to the YBG hierarchy previously mentioned [11],[17], and expresses a balance of forces: in the right-hand side of this equation there appear the mean force [17] and in the left-hand side the sum of two terms; the first is the force due to the van der Waals interactions with the  $n-1$  fixed particles different from the  $j$ -th particle in the  $n$ -th cluster (responsible for the compressions and dilatations of the  $n$ -th cluster), while the second which introduces long range correlations is the average force on the  $j$ -th particle from the remaining  $N-n$  particles of the system.

#### 4 Entropic oscillator

In Section 2, we have demonstrated through Eq. (15) that the distribution vector  $f(t)$  corresponding to the closed universe possesses an oscillatory behavior and in Section 3, we have established the relation between  $f(t)$  and the entropy  $S$  through Eq. (18). Therefore, as a consequence of these  $S(t)$  will also have an oscillatory behavior which can be described by

$$\frac{d}{dt} \left( \frac{\dot{S}(t)}{\sqrt{K_{eff}(t)}} \right) + \sqrt{K_{eff}(t)} S = 0, \quad (22)$$

if one assumes that  $S(t)$  behaves as a free oscillator with an "elastic" time dependent coefficient  $K_{eff}(t)$  defined through the relation

$$K_{eff}(t) = \frac{1}{S} \frac{\partial^2}{\partial t^2} S. \quad (23)$$

A characteristic value of  $K_{eff}(t)$  can be estimated from the consideration that the period of the entropy oscillations of the closed universe should be coherent with the Poincare cycles,  $\tau_p$ , which leads to

$$2\pi = \int_0^{\tau_p} \sqrt{K_{eff}(t)} dt. \quad (24)$$

From these considerations it follows that the entropy of the closed universe will oscillate between two "equilibrium" states:  $0 \geq S_{eq} \geq S \geq S_N + S_{eq}$ . This is one of the main results of this paper, since it strongly suggests that our statistical mechanical theory of a closed universe might complement the standard cosmological model of an oscillating universe based on purely general relativity grounds [1].

The implications of this theory on the concept of time can be extracted by first noticing that the dynamics (22) for the entropy of the universe corresponds the Hamiltonian [18]

$$\mathcal{H}(S, \dot{S}, t) = \frac{1}{2} \left( \frac{\dot{S}(t)}{\sqrt{K_{eff}(t)}} \right)^2 + \Phi(S), \quad (25)$$

where  $\Phi(S) = (1/2)(S - S^*)^2$ , with  $S^* = S_{eq} + S_N/2$ . This indicates that the central quantities in the statistical description are related to the entropy  $S$  and its rate of change  $\dot{S}$ . Moreover, since the solution of Eq. (22) is given by

$$S(t) - S^* = \Delta_S \cos[z(t) + \alpha], \quad (26)$$

where  $\Delta_S = S_N/2$ , and  $\alpha$  is a parameter containing the initial conditions, and where the rescaled time  $z(t)$  is defined through the relation

$$z(t) = \int \sqrt{K_{eff}(t')} dt', \quad (27)$$

the solution given through Eq. (26) of the Eq. (22) enables us to parametrize time in terms of the entropy through the relation

$$z(S) = \pm \frac{1}{\sqrt{2}} \int \frac{dt'}{\sqrt{\mathcal{H} - \Phi(S)}}. \quad (28)$$

For the potential  $\Phi(S)$  already introduced we explicitly obtain  $z(S) = \pm \sqrt{2} \arcsin \left[ (S - S^*) / \sqrt{2\mathcal{H}} \right]$ . From this relation we may conclude that time depends upon the amount of entropy generated by the universe during its evolution towards the final equilibrium state. More precisely, in view of Eq. (25), the quantity  $\sqrt{\mathcal{H} - \Phi(S)}$  constitutes the scaled entropy production and therefore time in the universe is self-generated by interactions through the entropy production. From this analysis it follows that during the evolution of the universe, time is non-homogeneous due to the non-homogeneous character of the entropy production of the universe itself.

## 5 Conclusions

In this article, we have proposed a statistical mechanic theory of a closed universe configured by physical interactions. These interactions define the lifespan of the universe itself and its volume in the configurational space. The whole universe can be represented as a point in a two-dimensional phase space determined by the entropy  $S$  and the entropy production  $\dot{S}$ . The universe evolves in its phase space along a closed trajectory which constitutes the phase portrait of a free entropic oscillator. Since the entropy production depends on the state of the universe through its instantaneous entropy, we conclude that time has a nonhomogeneous character.

The key point in our theory lies in the definition of the state of the universe by means of the distribution vector and the postulate of the Generalized Gibbs entropy as a functional of the distribution vector. Unlike the Gibbs entropy which is a constant of motion under the Liouville dynamics, the Generalized Gibbs entropy is a varying quantity whose variation is given by the Generalized Liouville dynamics.

Two main results follow from our analysis: The first one is that our theory predicts a closed oscillating universe, in accordance with the full microscopic description given by the Generalized Liouville equation, and that time is a consequence of dissipative interactions that may be quantified by the entropy production of the universe itself.

Quantum and relativistic effects can be taken into account by formulating our theory in terms of the set of reduced density operators and the corresponding generalization of the von Neumann equation with the suitable metric, which will be addressed in further works. As it was shown, since the main results of the present work are a consequence of the nature of the Liouville eigenvalues, despite of the details of these more convoluted mathematical formulations, the main conclusion of our model will still be valid.



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